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"Deco" polyominoes, permutations and random generation

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Abstract

In this paper, we introduce a class of polyominoes, called *deco polyominoes*, in bijection with the set of permutations of the first k integers. We evaluate some typical parameters for this class of polyominoes and define a linear algorithm for randomly generating them. Moreover, we define the polyominoes corresponding to the directed animals on the hexagonal and triangular lattices and examine these polyominoes' deco class. As far as deco polyominoes corresponding to animals on the hexagonal lattice are concerned, we find a bijection with a class of permutations. Finally, we give linear algorithms for randomly generating both hexagonal and triangular deco polyominoes.

1. Introduction

Polyominoes (as well as their equivalent animals) have been studied by mathematicians because of these structures' interesting combinatorial properties. Physicists have studied them because they act as models for various phenomena. If we take the $\mathbb{R} \times \mathbb{R}$ plane, we can define a *cell* as a unitary square $[i, i+1] \times [j, j+1]$ $(i, j \in \mathbb{N})$ and a *polyomino* as a connected set of cells' pairs having one side in common. The polyominoes are defined up to a translation. If we consider the centers of a polyomino's cells, we obtain the so called *animal* structure, which is totally equivalent to the polyomino from a combinatorial point of view. The definitions of the polyominoes' or animals' parameters are quite intuitive:

- a polyomino's *area* is the number of cells that make it up,
- a polyomino's *perimeter* is the number of edges of its border,
- a *column* (resp. *row*) is the intersection of a polyomino with an infinite vertical (resp. horizontal) strip $[i, i+1] \times \mathbb{R}$ (resp. $\mathbb{R} \times [j, j+1]$).

The concept of direction and convexity can be applied to both polyominoes and animals. We can obtain a directed polyomino by starting out from a cell called *source* and by adding other cells in predetermined directions, such as East and North, that is, to the right of or over existing cells. In this way, a polyomino grows in a so-called

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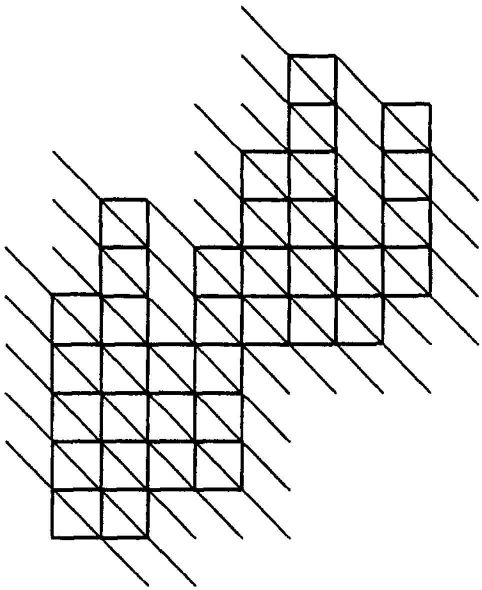

Fig. 1. A directed column-convex polyomino.

privileged direction. A polyomino is said to be *column-convex* (resp. *row-convex*) if all its columns (resp. rows) are connected. A *convex* polyomino is a polyomino which is both row- and column-convex. A directed polyomino's height is the number of lines perpendicular to the privileged direction that go through the cell centers (see Fig. 1). If we make the polyomino rotate so that the privileged direction coincides with the North–South direction, its height indicates the number of cell center levels. This definition coincides with the one for binary trees and other structures. Even though polyominoes seem to be simple structures, their combinatorial properties are quite difficult to understand. For example, the problem of enumerating directed animals according to their area was initially solved six years ago by Gouyou-Beauchamps and Viennot [12]; subsequently, Penaud [14] solved it in a simpler way. A lot of problems have been solved by finding some bijections [9] between the various objects studied and the words of some algebraic language and applying the DSV-methodology [15]. This methodology allows us to obtain some generating functions from a non-ambiguous grammar. Several classes of polyominoes have been enumerated according to their area, perimeter, number of columns, etc., or a combination of these parameters [6–8, 18]. Particular attention has been given to studying the most significant parameters in some models of physical phenomena. However, it was not possible to establish theoretical, exact or asymptotic evaluations for all of the parameters studied. As a result, random generation algorithms are of fundamental importance in the various classes of polyominoes. Thanks to these algorithms, it is possible to generate a suitable number of objects having a predefined size and thus obtain an adequate sample to examine experimentally. This gives us a double advantage: we are able to check (or reject) the conjectures and we get information about the parameters we have no theoretical results for. Random generation algorithms efficiency is obviously very important because we have to generate a great many structures, some very large, in order to obtain significant experimental data.

It is worth noting that a random generation algorithm that is efficient for a certain structure class is not usually efficient for its subclasses. An example of this is the linear random generation algorithm for directed animals [4]: it cannot be applied to the subclass of column-convex polyominoes [3].

2. Deco Polyominoes

The *average height* (called *length*, by some authors) of directed animals is particularly important when they are used as models of physical phenomena because their height is connected to some basic parameters [17]. Nonetheless, no exact or asymptotic evaluations have been found, while a conjecture has been made by some physicists [10], which has not been contradicted by the random generation of predetermined area animals [2]. In [1, 3], we studied directed column-convex polyominoes for two main reasons:

(i) we wanted to verify the convexity's influence on typical parameters,

(ii) we were able to obtain more theoretical results for this class than for the larger class made up of directed polyominoes.

By using some recurrence relations, we succeeded in defining a linear algorithm for randomly generating directed column-convex [3] polyominoes, counting them according to their area and height and determining the average height h_n [1]:

$$h_n \approx \frac{\phi}{\sqrt{5}}\, n + 0.87647957778,$$

in which $\phi = (1+\sqrt{5})/2$. (Naturally, we also obtained some known results in a very simple way.) Moreover, we obtained an expression for the number W_k of directed column-convex polyominoes of height k [1]:

$$W_k = k!\, k \left(1 - \sum_{j=1}^{k-1} \frac{1}{j\,(j+1)\,(j+1)!}\right). \tag{1}$$

This formula is the solution of the following recurrence:

$$W_{k+1} = (k+1)\, W_k + \sum_{j=1}^{k} W_j, \tag{2}$$

which can be obtained in both a combinatorial way and as shown in [1].

Unfortunately, we do not have a direct combinatorial interpretation of the formula (1) but there is a factorial in it and this suggests that there might be a class of polyominoes in bijection with the permutations and therefore counted by $k!$. As will be shown further on, we introduce this class and define it as the set of directed

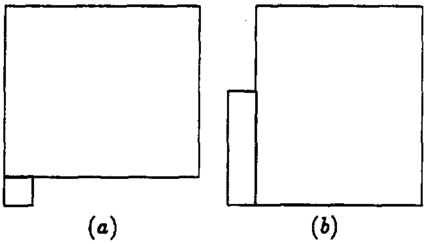

Fig. 2. Decomposition of the deco polyominoes.

column-convex polyominoes having height k and which get the height k only in the last column. We call these polyominoes *deco polyominoes* after the French *de̲rnière co̲lonne*: last column. In order to prove that the number D_k of deco polyominoes of height k is $k!$, we define a recurrence relation based on a decomposition of these polyominoes. Let us examine the right side of the source: if there is no cell there, the polyomino is made up of the source and a deco polyomino of height $k-1$ attached to the North side (see Fig. 2(a)) and therefore there are D_{k-1} possibilities. Vice versa, when there is a cell on the source's right, the polyomino is made up of the first column containing fewer than k cells (because height k is only obtained in the last column) and a deco polyomino of height $k-1$ attached to the East side of the source (see Fig. 2(b)). Consequently, there are $(k-1)\,D_{k-1}$ possibilities and we obtain the relation $D_k = kD_{k-1}$ and the condition $D_1 = 1$, whose solution is $D_k = k!$. Each deco polyomino of height $k-1$ is used k times in this type of construction. Fig. 3 illustrates the deco polyominoes of height 4 and their decomposition.

The decomposition suggests the following bijection between permutations and deco polyominoes. Let $(a_1, a_2, \ldots, a_k)$ be a permutation of the first k integers. We build the corresponding polyomino recursively. The single-cell polyomino corresponds to the permutation only made up of $a_1 = 1$. If $k > 1$ and $a_1 = k$ the polyomino is made up of the source and the deco polyomino of height $k-1$ corresponding to the permutation $(a_2, a_3, \ldots, a_k)$ and that is attached to the North side of the source (see Fig. 2(a)). If $a_1 = j < k$, the polyomino is made up of a column of j elements and the deco polyomino of height $k-1$ corresponding to the permutation $(b_2, b_3, \ldots, b_k)$ defined as follows:

- if $a_i < j$ then $b_i = a_i$,
- if $a_i > j$ then $b_i = a_i - 1$.

$(b_2, b_3, \ldots, b_k)$ is a permutation of the first $k-1$ integers and the corresponding polyomino is joined to the East side of the source (see Fig. 2(b)).

This bijection allows us to define an algorithm for the random generation of deco polyominoes. As a matter of fact, we can use a linear algorithm for the random generation of permutations and then build the corresponding polyomino by means of the above-mentioned bijection. We can say, however, that this algorithm is not linear

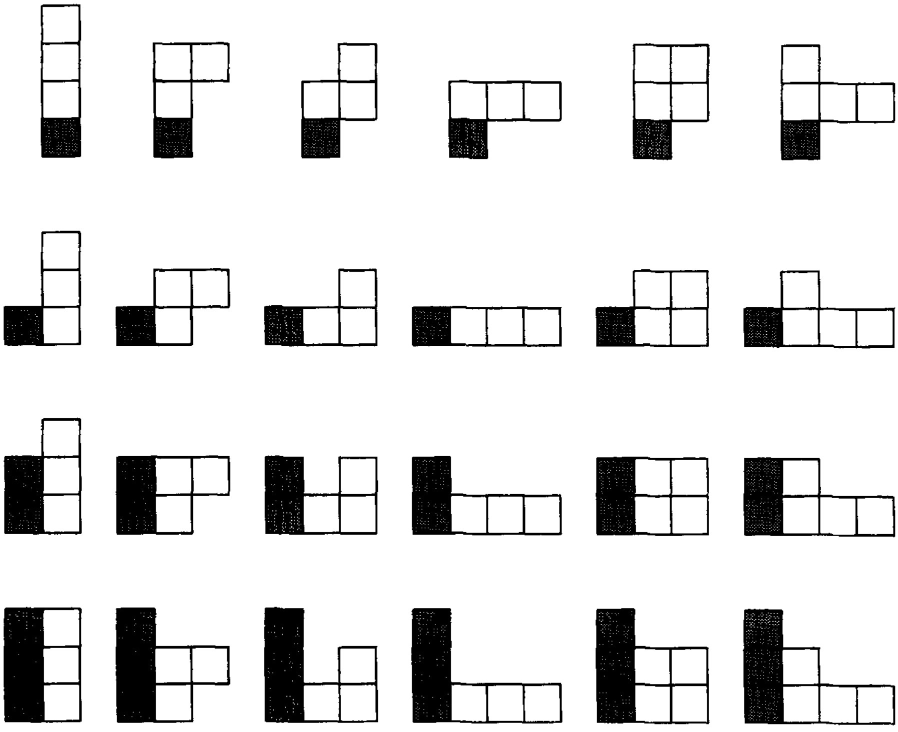

Fig. 3. Deco polyominoes of height 4.

because of the checks and changes made at every step on the part of the permutation to be used in the subsequent steps.

In order to find a linear algorithm, we remark that a deco polyomino of height k can also be coded by means of a sequence $(c_k, c_{k-1}, \ldots, c_1)$ such that $1 \leqslant c_j \leqslant j$, $1 \leqslant j \leqslant k$. The recursive procedure that constructs the polyomino corresponding to a given sequence is analogous to the previous one:

- if $c_k = k$ then the polyomino is made up of a source and the polyomino corresponding to the sequence $(c_{k-1}, \ldots, c_1)$ and that is attached to the North side of the source;
- if $c_k = j < k$ then the polyomino is made up of a first column of height j and the polyomino corresponding to the sequence $(c_{k-1}, \ldots, c_1)$ and that is joined to the East side of the source.

This procedure does not modify the sequence to be used in the subsequent steps. We can therefore define a random generation linear algorithm that produces a sequence $(c_k, c_{k-1}, \ldots, c_1)$ from which it builds up the polyomino. The following is the Pascal description of the algorithm: each column is represented by a pair (c, d) in which c indicates the number of cells and d the displacement of the subsequent column; *append* is a simple routine appending the pairs (c, d) to a global list L which will eventually contain the result of the generation.

```
Procedure deco (k : integer);
var c, d, i, r : integer;
begin
    c := 0; d := 0;
    for i := k downto 2 do
        begin
            r := random(i) + 1;
            if r = i then
              begin c := c + 1; d := d + 1 end
            else
              begin c := c + r; append(L, c, d);
                    c := 0; d := 0 end
        end;
    c := c + 1;
    append(L, c, 0)
end;
```

The linearity of this algorithm is obvious. By again using some recurrence relation we obtain some other theoretical results, such as the average area and average number of columns.

With regard to the total area A_k of all deco polyominoes of height k, we again refer to Fig. 2 and get the following relation:

$$A_k = k\,A_{k-1} + D_{k-1} + \frac{k(k-1)}{2}\,D_{k-1}.$$

By dividing it by $D_k = k!$, we obtain a relation for the average area $a_k = A_k/D_k$:

$$a_k = a_{k-1} + \frac{1}{k} + \frac{k-1}{2},$$

and the condition $a_1 = 1$.

Therefore,

$$a_k = \frac{k(k-1)}{4} + H_k,$$

in which $H_k = \sum_{j=1}^{k} 1/j$.

In the same way, for the total number of columns, we have

$$C_k = k\,C_{k-1} + (k-1)\,D_{k-1},$$

and for the average number $c_k = C_k/D_k$,

$$c_k = c_{k-1} + \frac{k-1}{k},$$

and the condition $c_1 = 1$, and therefore

$$c_k = k + 1 - H_k.$$

We again consider the decomposition illustrated in Fig. 2 in order to show a connection between deco polyominoes and Stirling numbers of the first kind. We denote by $D_{k,j}$ the number of deco polyominoes having height k and j columns; we have the following relation:

$$D_{k,j} = D_{k-1,j} + (k-1)\,D_{k-1,j-1}$$

and the conditions

$$D_{k,1} = 1, \quad k \geq 1,$$

$$D_{k,j} = 0, \quad k < j \quad \text{or} \quad j < 1.$$

If we set $D'_{k,k-j+1} = D_{k,j}$, we obtain

$$D'_{k,k-j+1} = D'_{k-1,k-j} + (k-1)\,D'_{k-1,k-j+1}.$$

We can rewrite this relation as

$$D'_{k,i} = D'_{k-1,i-1} + (k-1)\,D'_{k-1,i}$$

with the conditions

$$D'_{k,k} = 1, \quad k \geq 1,$$

$$D'_{k,i} = 0, \quad i < 1 \quad \text{or} \quad i > k.$$

This is the relation defining the Stirling numbers of the first kind (see [13]); therefore, we obtain

$$D_{k,j} = D'_{k,k-j+1} = s(k, k-j+1).$$

Since $D_k = \sum_{j=1}^{k} D_{k,j}$, we get a simple combinatorial proof of

$$k! = \sum_{j=1}^{k} s(k,j).$$

3. Deco polyominoes on other lattices

There is a well-known correspondence between animals on square lattices and polyominoes having square cells. Since animals have also been defined on triangular and hexagonal lattices [11] (see Fig. 4), we are going to consider the corresponding polyominoes. However, in this case, it is not possible to use square cells whose centers correspond to the animal's nodes because there could be cells attached to each other

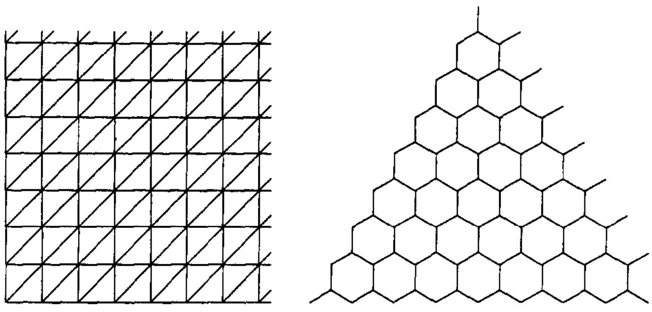

Fig. 4. Triangular and hexagonal lattices.

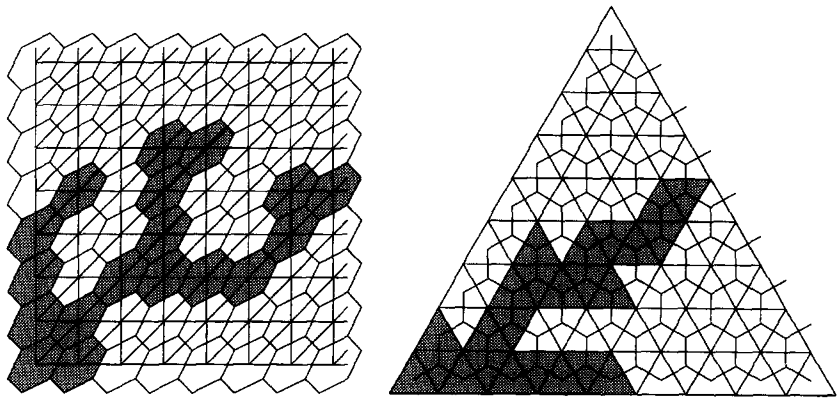

Fig. 5. Hexagonal and triangular polyominoes.

at a vertex on a triangular lattice and spaces left between cells that should be adjacent on a hexagonal lattice. It is therefore necessary to refer hexagonal cells to a triangular lattice and triangular cells to a hexagonal lattice (see Fig. 5). In this way, we can also define the concept of column and vertical convexity (column-convex) and establish recurrence relations similar to the ones used for the square lattice.

3.1. *Triangular deco polyominoes*

Let us now examine deco polyominoes having triangular cells. As far as the source is concerned, we can choose between two types of cells (see Fig. 6) which correspond to animal nodes with an outdegree of 1 or 2. In the first case, it is easier for us to define some recurrences. On the other hand, every triangular polyomino whose source is of the second kind corresponds to a polyomino having a source of the first kind, plus another cell. From now on we will use only sources of the first kind. As was the case for square deco polyominoes, we have to take two different

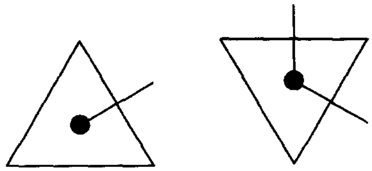

Fig. 6. Sources of the triangular polyominoes.

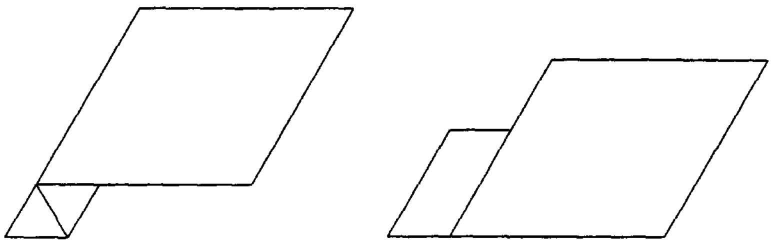

Fig. 7. Decomposition of the triangular deco polyominoes.

cases into account when we want to obtain a recurrence for the number T_k of triangular deco polyominoes of height k. In this case however, if a polyomino has more than one cell, there must also be the cell adjacent to the source. This cell is part both of the first row and the first column and the subsequent column is attached to it.

Let us now examine the first row: if it only contains the source and the adjacent cell, then the deco polyomino of height k is made up of these two cells and a deco polyomino of height $k-2$ (see Fig. 7). If, on the contrary, the first row contains three cells or more, then the deco polyomino of height k is made up of the first column (that must contain at least two cells and at most $k-1$) and a polyomino of height $k-2$ attached to the right of the cell adjacent to the source. We therefore obtain the following relation:

$$T_k = (k-1)\,T_{k-2},$$

and the conditions $T_1 = 1$ and $T_2 = 1$, from which we obtain the expression

$$T_k = (k-1)(k-3)(k-5)\ldots 1,$$

which in the turn can be written in two different ways according to whether k is even or not:

$$\begin{cases} T_{2n} = \dfrac{(2n)!}{2^n\, n!}, \\ T_{2n+1} = 2^n\, n!. \end{cases}$$

It is also possible to obtain the average area a_k and the average number of columns c_k for triangular deco polyominoes of height k:

$$\begin{cases} a_{2n} = \dfrac{n(n+1)}{2} + H_{2n} - \dfrac{H_n}{2}, \\ a_{2n+1} = \dfrac{n(n+2)}{2} + 1 + \dfrac{H_n}{2}, \end{cases}$$

$$\begin{cases} c_{2n} = (n+1) - H_{2n} + \dfrac{H_n}{2}, \\ c_{2n+1} = (n+1) - \dfrac{H_n}{2}. \end{cases}$$

We can also define a bijection between these polyominoes and a class of permutations. We refer to permutations in which every odd-place element is higher than all the following ones. We build a polyomino from a permutation in the same way as we build a square deco polyomino.

It is worth noting that there is only one possibility for attaching a cell (or a column of j cells) to a cell of the first type. This agrees with the fact that these cells correspond to odd-place permutation elements. In this case, too, we are able to define a random generation algorithm based on the bijection, but the algorithm would not be linear. We can represent this class of polyominoes too by means of the sequences $(c_k, c_{k-1}, \ldots, c_1)$ such that $1 \leq c_j \leq j$, $1 \leq j \leq k$, moreover, in this case we have to impose the condition $c_{k-2i} = k - 2i, i = 1, \ldots, \lfloor k/2 \rfloor$. We follow the same process used for square polyominoes and obtain the following linear algorithm for generating triangular deco polyominoes of height k:

```
Procedure tdeco (k : integer);
var c, d, i, r : integer;
begin
    c := 0; d := 0;
    for i := 1 to k - 1 do
        if odd(i) then
            begin c := c + 1; d := d + 1 end
        else
            begin
                r := random(k - i + 1) + 1;
                if r = k - i + 1 then
                    begin c := c + 1; d := d + 1 end
                else
                    begin c := c + r; append(L, c, d);
                          c := 0; d := 0 end
            end;
    c := c + 1;
    append(L, c, 0)
end;
```

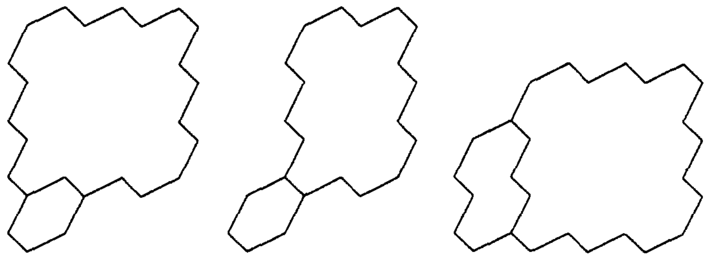

Fig. 8. Decomposition of the hexagonal deco polyominoes.

3.2. Hexagonal deco polyominoes

Let us now examine hexagonal deco polyominoes. It is possible to attach other cells to each cell in three different directions: North, East and Northeast. We must take three different cases into account in order to define a recurrence relation for the number E_k of hexagonal polyominoes of height k (see Fig. 8). If there is no cell to the East of the source but there is one to the North of it, then the polyomino is made up of the source and a hexagonal deco polyomino of height $k-1$ attached to the North side of the source. If there is no cell to the East and to the North of the source, then the polyomino is made up of the source and a hexagonal deco polyomino of height $k-2$ attached to the Northeast of the source. Finally, if there is a cell to the East of the source, then the polyomino is made up of the first column (which can contain up to $k-1$ elements) and a deco polyomino of height $k-1$ attached to the East of the source. We therefore obtain the relation

$$E_k = k\,E_{k-1} + E_{k-2},$$

and the conditions $E_1 = 1$, $E_2 = 2$ (Sequence 704 in Sloane's book [16]). From this relation we were not able to find an explicit expression for E_k but, by following Birkhoff [5] and by using Maple, we succeeded in getting an approximate one:

$$E_k = \frac{c}{\sqrt{2\pi}}\,k!\,\left(1 - \frac{1}{k} + \frac{1}{2k^2} + \mathrm{O}(k^{-3})\right), \tag{3}$$

in which c is a constant whose estimate is $c \approx 3.987$.

In order to find an expression for E_k we take a second parameter into consideration. We say that there is a *diagonal step* in an hexagonal deco polyomino whenever a column is attached to the Northeast side of the previous one. We denote by $E_{k,p}$ the number of hexagonal deco polyominoes having height k and p diagonal steps. From the construction shown in Fig. 8 we can easily obtain the following relation:

$$E_{k,p} = k\,E_{k-1,p} + E_{k-2,p-1}, \tag{4}$$

and the conditions

$$E_{k,0} = k!, \quad k > 0,$$
$$E_{k,p} = 0, \quad p > \left\lfloor \frac{k-1}{2} \right\rfloor$$

which arise from the fact that hexagonal polyominoes without diagonal steps correspond to square polyominoes. By defining $E'_{k-p,p+1} = E_{k,p}$ we can rewrite relation (4) as

$$E'_{k-p,p+1} = k\, E'_{k-p-1,p+1} + E'_{k-p-1,p}$$

and the conditions

$$E'_{j,1} = j!, \quad j > 0$$

This is the relation for Lah's numbers [13]; we therefore obtain

$$E_{k,p} = E'_{k-p,p+1} = \frac{(k-p)!}{(p+1)!} \binom{k-p-1}{p},$$

and the following expression for E_k:

$$E_k = \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} E_{k,p} = \sum_{p=0}^{\lfloor (k-1)/2 \rfloor} \frac{(k-p)!}{(p+1)!} \binom{k-p-1}{p}. \tag{5}$$

From this we deduce that the constant in the expansion (3) of E_k is

$$\frac{c}{\sqrt{2\pi}} = \sum_{p=0}^{\infty} \frac{1}{p!(p+1)!} = I_1(2),$$

where I_1 is the modified Bessel function of the first kind and of order 1 [13].

In order to represent hexagonal deco polyominoes of height k by means of sequences composed by k integers, we point out that diagonal steps are the sole difference between hexagonal and square polyominoes. We can think of a diagonal step as it was obtained by adding a *ghost-cell* to the North side of a column and a polyomino to the East side of the ghost-cell.

Therefore an hexagonal deco polyomino of height k can be coded by means of a sequence $(c_k, c_{k-1}, \ldots, c_1)$ such that $c_1 = 1$, $0 \leq c_j \leq j$ $(1 < j < k)$, $1 \leq c_k \leq k$ and $c_{j+1} = j+1$ if $c_j = 0$.

The procedure building up a polyomino from a sequence is similar to that in the square case except for the diagonal steps (or ghost-cells):

- if $c_k = k$ and $c_{k-1} \neq 0$ then the polyomino is made up of a source and the polyomino corresponding to the sequence $(c_{k-1}, \ldots, c_1)$ and that is attached to the North side of the source;
- if $c_k = k$ and $c_{k-1} = 0$ then the polyomino is made up of a source and the polyomino (of height $k-2$) corresponding to the sequence $(c_{k-2}, \ldots, c_1)$ and that is joined to the Northeast side of the source. The pair $(k, 0)$ corresponds to a diagonal step;

- if $c_k = j < k$ then the polyomino is made up of a column having j cells and the polyomino corresponding to the sequence $(c_{k-1}, \ldots, c_1)$ and that is joined to the East side of the source.

As far as random generation is concerned we use again the bijection between polyominoes and sequences. However, we must take into account the fact that the values in the domain of any c_j are not equiprobable. In fact, we have

$$Pr(c_k = j) = \frac{E_{k-1}}{E_k}, \quad 1 \leq j < k$$

and

$$Pr(c_k = k) = \frac{E_{k-1} + E_{k-2}}{E_k}$$

or, equivalently,

$$Pr(c_k = j \wedge c_{k-1} \neq 0) = \frac{E_{k-1}}{E_k}, \quad 1 \leq j \leq k$$

and

$$Pr(c_k = k \wedge c_{k-1} = 0) = \frac{E_{k-2}}{E_k}.$$

The following Pascal procedure shows a possible implementation of the algorithm for the hexagonal deco polyomino random generation.

```
Procedure hdeco (k : integer);
var c,d,j,r : integer;
    p : real;
begin
    c := 0; d := 0; j := k
    while j ≥ 2 do
        begin
            p := random;
            if p ≥ j * tab[j − 1]/tab[j] then {c_j = j ∧ c_{j−1} = 0}
              begin append(L,c + 1,d + 1);
                    c := 0; d := 0; j := j − 2  end
            else
                begin
                    r := random(j) + 1;
                    if r = j then {c_j = j ∧ c_{j−1} ≠ 0}
                      begin c := c + 1; d := d + 1; j := j − 1  end
                    else {c_j < j}
                        begin c := c + r; append(L,c,d);
                              c := 0; d := 0; j := j − 1  end
                end
        end
    c := c + 1;
    append(L,c,0)
end;
```

A pair (c, c) means that the corresponding column contains c elements and the following column is attached to the Northeast side of the highest cell. This procedure uses the global variable *tab* which is an array containing values E_j; this array can be built up, once forever, in linear time by using the recurrence relation. It is therefore evident that the algorithm performs linearly with respect to time and space.

Acknowledgements

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